13.1 + 13.2. Vector functions and curves

Def (1) A <u>vector function</u> is a function whose outputs are vectors.

(2) A <u>curve</u> is an object parametrized by a vector function of one variable.

e.g. $\vec{\ell}(t) = (2+t,\ 3-2t,\ 1+2t) \rightsquigarrow$ a line

$\vec{r}(t) = (\cos t,\ \sin t,\ 0) \rightsquigarrow$ a circle

$(\because x^2+y^2=1,\ z=0)$

Prop Consider a vector function $\vec{r}(t) = (f(t), g(t), h(t))$.

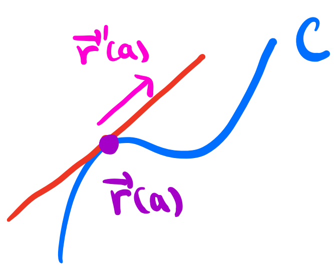
(1) $\lim_{t \to a} \vec{r}(t) = \left(\lim_{t \to a} f(t),\ \lim_{t \to a} g(t),\ \lim_{t \to a} h(t) \right)$.

(2) $\vec{r}\,'(t) = (f'(t),\ g'(t),\ h'(t))$.

(3) $\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt,\ \int_a^b g(t)\,dt,\ \int_a^b h(t)\,dt \right)$.

(4) $\int_a^b \vec{r}\,'(t)\,dt = \vec{r}(b) - \vec{r}(a)$.

Prop If a curve C is parametrized by $\vec{r}(t)$, then the tangent line to C at $\vec{r}(a)$ has a direction vector $\vec{r}\,'(a)$.



$\rightsquigarrow$ $\vec{r}\,'(a)$ is the tangent vector at $\vec{r}(a)$.

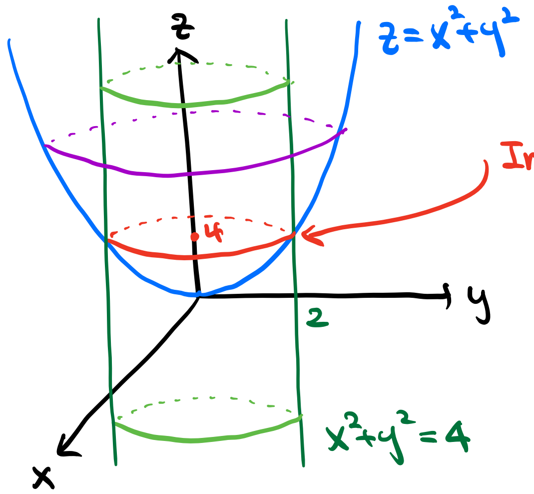
<u>Ex</u> Find a vector function which parametrizes the intersection of the surfaces $x^2+y^2=4$ and $z=x^2+y^2$.

<u>Sol</u> Solve the system $x^2+y^2=4$ and $z=x^2+y^2$

$$\Rightarrow \begin{cases} z = x^2+y^2 = 4 \\ x^2+y^2=4 \rightsquigarrow x=2\cos t,\ y=2\sin t \text{ with } 0\le t\le 2\pi. \end{cases}$$

$$\rightsquigarrow \boxed{\vec{r}(t) = (2\cos t,\ 2\sin t,\ 4) \text{ with } 0\le t\le 2\pi.}$$

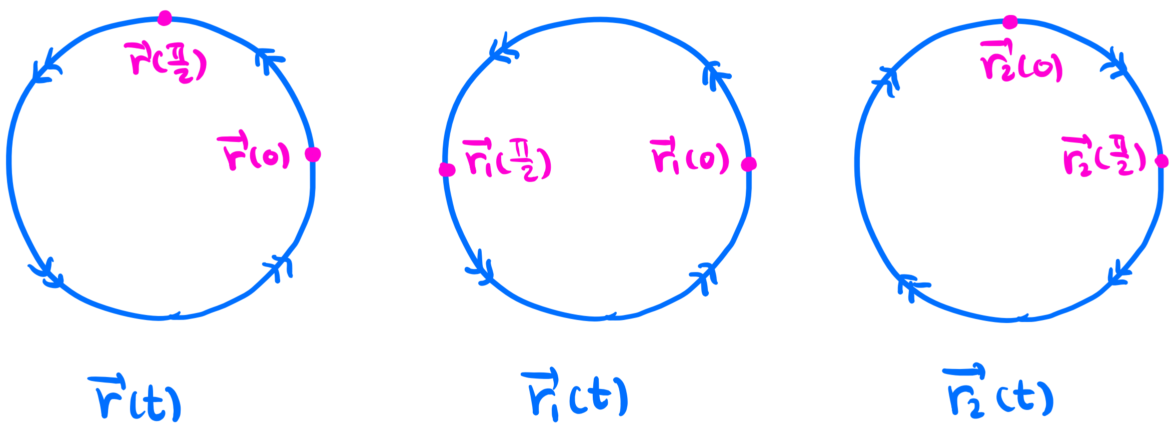
<u>Note</u> (1) You can see this from a sketch:



(2) There are other parametrizations:

$$\vec{r}_1(t) = (2\cos(2t),\ 2\sin(2t),\ 4) \text{ with } 0\le t\le \pi.$$

$$\vec{r}_2(t) = (2\sin t,\ 2\cos t,\ 4) \text{ with } 0\le t\le 2\pi.$$



Ex Sketch the curve parametrized by each function.

(1) $\vec{r}(t) = (\cos t, \sin t, t)$

Sol Idea: Find a surface which the curve must lie on by finding a relation between the coordinate functions.

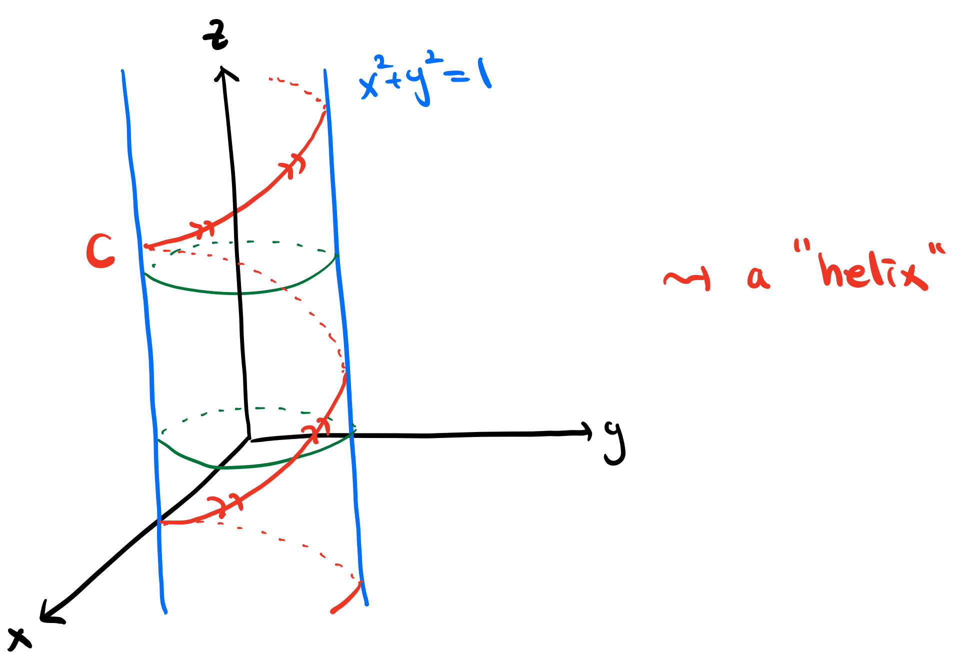
For $\vec{r}(t)$: $x = \cos t$, $y = \sin t$, $z = t$

$\rightsquigarrow x^2 + y^2 = \cos^2 t + \sin^2 t = 1$.

$\Rightarrow$ The curve must lie on the cylinder $x^2+y^2=1$.

Also, $z=t$ increases as t increases.

$\Rightarrow$ The curve spirals upward along the cylinder.



Note $x = \cos t$ and $y = \sin t$ together yield a rotation around the z-axis, while $z=t$ moves upward.

(2) $\vec{r}(t) = (t\cos t,\ t^2,\ t\sin t)$ with $t \geq 0$.

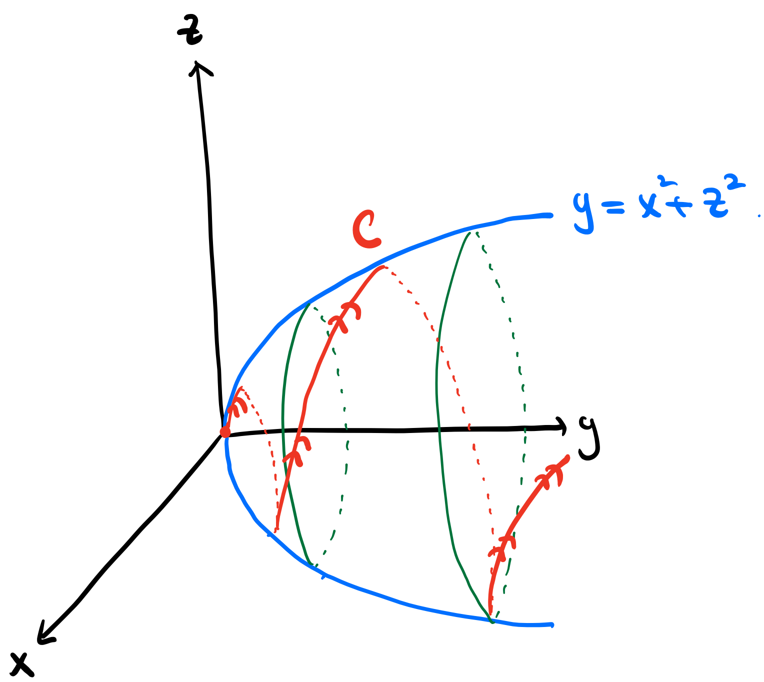
<u>Sol</u> For $\vec{r}(t)$: $x = t\cos t$, $y = t^2$, $z = t\sin t$.

$\rightsquigarrow x^2 + z^2 = t^2\cos^2 t + t^2\sin^2 t = t^2 = y$

$\Rightarrow$ The curve lies on the paraboloid $y = x^2 + z^2$.

Also, $y = t^2$ increases as t increases.

$\Rightarrow$ The curve spirals around the paraboloid $y = x^2 + z^2$ in the positive y-direction.



<u>Note</u> $x = t\cos t$ and $z = t\sin t$ together yield a rotation around the y-axis with increasing radius, while $y = t^2$ moves to the positive y-direction.

Ex Let C be the curve parametrized by

$$\vec{r}(t) = (t^2, 3t-1, t+1).$$

Parametrize the tangent line to C at $(4,5,3)$.

Sol Find the value of t at $(4,5,3)$:

$$\vec{r}(t) = (4,5,3) \rightsquigarrow (t^2, 3t-1, t+1) = (4,5,3).$$

$$\Rightarrow t^2 = 4, \quad 3t-1 = 5, \quad t+1 = 3$$

$$\Rightarrow t = 2$$

$*$ $t=-2$ works only for the first equation.

Find the tangent vector:

$$\vec{r}\,'(t) = (2t, 3, 1) \rightsquigarrow \vec{r}\,'(2) = (4, 3, 1)$$

$\Rightarrow$ The tangent line at $(4,5,3)$ is parametrized by

$$\boxed{\vec{l}(t) = (4+4t, 5+3t, 1+t)}$$